Circular motion

Isaac Newton[a] and Richard Conn Henry[b]

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An extraordinarily simple and transparent derivation of the formula for the acceleration that occurs in uniform circular motion is presented, and is advocated for use in high school and college freshman physics textbooks.

The familiar formula for the acceleration that occurs in uniform circular motion,

$$a=\frac{v^2}{r}, \tag{1}$$

may be, today, a staple of freshman physics, but just 350 years ago it represented the cutting edge of physics research. Its introduction marked the real beginning of the mathematization of physics. How did it arise? And most important, how should it be taught to students today?

The recognition that uniform circular motion does involve acceleration began with Galileo and Descartes; however, ''their handling of the problem was remarkably vague, and their qualitative, sometimes fuzzy, explanations of circular motion allowed them to ignore basic flaws in their respective systems of the world.'' [1]

The person who *solved* the problem was Christiaan Huygens in his book *Horologium Oscillatorium* (1658; second edition in 1673), the actual proof only appearing in Huygens' posthumous book *De Vi Centrifuga* (1703). The effect of his discovery was immense: Edmund Halley, Christopher Wren, and Robert Hooke were all able immediately to substitute the result into Kepler's third law and deduce that the gravitational force must vary inversely as the square of the distance from the sun. Their method produces the greater part of the law of gravitation, as I now show.

Consider two planets, masses m and M, that are in circular orbits and are distant from the sun r and R, and have years of length t and T. According to Kepler's third law,

$$\frac{T^2}{t^2}=\frac{R^3}{r^3}, \tag{2}$$

while Huygens provided

$$\frac{f}{F}=\left(\frac{mv^2}{r}\right)\left(\frac{R}{MV^2}\right)$$

$$=\frac{mRv^2}{MrV^2}=\frac{mR}{Mr}\,\frac{4\pi^2r^2}{t^2}\,\frac{T^2}{4\pi^2R^2}=\frac{mrT^2}{MRt^2} \tag{3}$$

(where I have also used the definition of the period, $t=2\pi r/v$). Combining these two results gives

$$\frac{f}{F}=\frac{mrR^3}{MRr^3}=\frac{(m/r^2)}{(M/R^2)}, \tag{4}$$

which yields $f\propto m/r^2$ or $f=GM(m/r^2)$ using Newton's later value for the constant of proportionality. Given this, is it any wonder that Hooke felt, to the end of his days, that *he* had discovered the law of gravitation? But while none of the three could deduce Kepler's laws from their result, Newton could and did. Halley and Wren, properly recognizing the genius of Newton, never made any claim of discovery.

Which, indeed, brings us to Newton. Did Newton know of Huygens' formula? Hooke, Halley, and Wren did, but Newton was intellectually more isolated in Cambridge, which in those days was a backwater compared with London. Perhaps it still is![2] Newton did obtain a copy of *Horologium*, from Henry Oldenburg, shortly after its publication. In any case, Newton independently obtained the formula, and in a way that is *much superior* to that employed by Huygens.

Huygens' method will be discussed briefly below, as a matter of historical interest. The main aim of my article, however, is to present *Newton's* method *so that it can be used in effectively teaching the origin of Eq. (1) to the modern young student of physics*.

The essential problem that all the early physicists faced was the need to work a vectorial problem, not having a real theory of vectors available. What *was* available to them was the parallelogram law for combining vectors in two-dimensional problems; that result is as old as Aristotle.[3]

If one *does* have vector methods, there are modern methods for obtaining Eq. (1) that are delightful.[4] But Eq. (1) is so crucial that one does not want the beginning physics student to think that it cannot be obtained except by sophisticated methods, with which the student is only beginning to become familiar. That is one reason why a clear exposition of Newton's wonderful method is pedagogically valuable.

Newton's Method. The method used by Newton, about 1665, is described by Westfall,[5] and has been analyzed in detail, from an historical perspective, by Erlichson.[6] My aim, here, however, is not historical, it is to present the method as it can and should be presented to the modern student.

The starting point is the resolution of forces into components (parallelogram law). As we shall see, that is really all that is needed; and that is nice, because teaching students to resolve forces into components is of course basic and necessary in any case, and the present application gives the students a tremendous reward, quickly, for mastering the notion.

In Fig. 1(a), I show a particle that is moving in a straight line with constant speed v and which will soon hit a hard surface at **A**. The component of the velocity $\mathbf{v}$ that is perpendicular to the surface is $v\cos\theta$. In Fig. 1(b), the particle has undergone the impact, and the component of the velocity that is perpendicular to the surface is again $v\cos\theta$, but it is now directed oppositely, so that the *total change* in velocity is

$$\Delta v=2v\cos\theta \tag{5}$$

(the component of the velocity that is parallel to the surface, $v\sin\theta$, not having changed in the impact). It is in the change of *direction* that the change of *velocity* enters. And the change in velocity, we see, is in a direction that is *normal* to the surface struck.

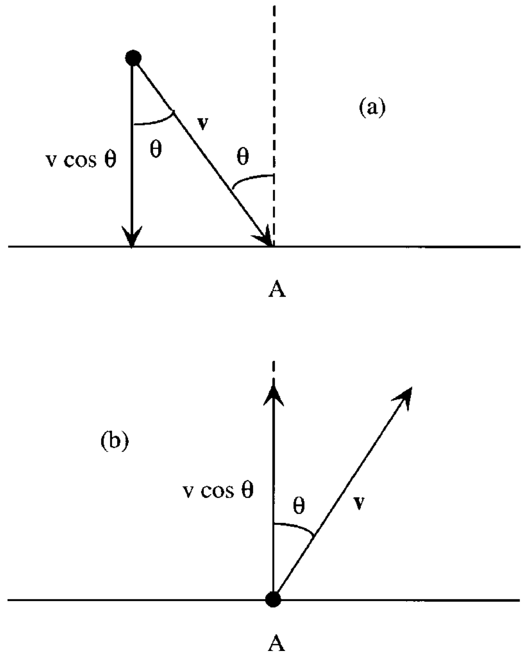

Fig. 1. A particle moving with velocity **v** strikes a hard surface at *A*, rebounding at the same angle θ with which it struck. The component of its velocity in the horizontal direction is always $\nu \sin \theta$, and the particle always moves with *speed* v.

And *that is it!* All we need to do now, is to *apply* this result, as Newton did, to the case of a particle that is crashing around a circle (Fig. 2). The figure shows the case ''circle traversed in $n=12$ impacts,'' but of course Newton and I are going to let n go to infinity and φ go to zero very soon.

It should be emphasized that what is important is the *change* in velocity, not *what produces* that change. Whether the change is produced by a crash with a wall, or by gravitation, the change in the velocity is the change in the velocity.

The first point we note is that the *change* in velocity is *always* directed toward the center of the circle (i.e., is always normal to the surface struck) at each impact.

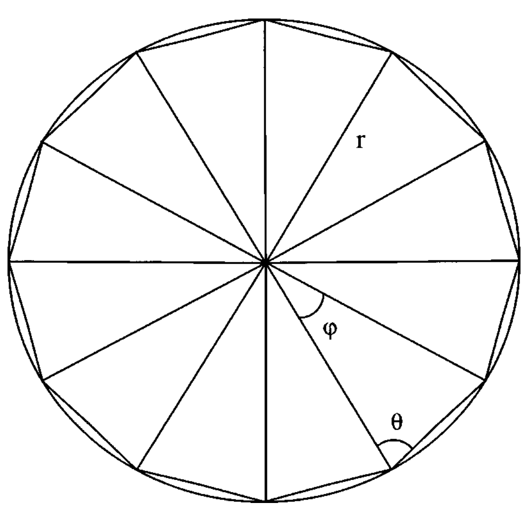

Fig. 2. A particle bounces around a circle, hitting the circle a total of 12 times, and each time bouncing at an angle θ to the normal to the circle.

We have seen that the change in velocity due to *each* impact is $\Delta v = 2v \cos \theta$, so the *sum* of the fractional changes that result from the n impacts is

$$\Sigma \Delta v = 2nv \cos \theta. \tag{6}$$

But $n\varphi = 2\pi$ and $\varphi + 2\theta = \pi$ so

$$\Sigma \Delta v = 2\,\frac{2\pi}{\varphi}\, v \cos\left(\frac{\pi}{2} - \frac{\varphi}{2}\right) = \frac{4\pi}{\varphi}\, v \sin\left(\frac{\varphi}{2}\right)$$

$$\cong \frac{4\pi}{\varphi}\, v\, \frac{\varphi}{2} = 2\pi v, \tag{7}$$

where in the penultimate step I have recognized that for small angles the sine is approximately the angle; when n is infinity, and φ is zero, our result is exact.

So, we conclude that $\Sigma \Delta v = 2\pi v$ in going around a complete circle, whatever may be the cause of the circular motion.

It should be pointed out again to the students that although the speed of the particle never changes, always being simply v, because of the change of *direction* in going around the circle, the velocity *does* change, every time the particle bounces.

We are now in a position to compute the acceleration, which is what we want. Because all the bounces are the same, we can calculate the acceleration *either* from the velocity change in any one bounce, *or* (as Newton did) from the sum of the fractional changes. Newton's approach gives

$$a = \frac{\Delta v}{\Delta t} = \frac{\Sigma \Delta v}{T}\; \frac{2\pi v}{\left(\dfrac{2\pi r}{v}\right)} = \frac{v^2}{r}, \tag{8}$$

where T is the time to go around the circle. So we have our desired result!

We have already noted that the change in velocity is always in the direction perpendicular to the surface struck, that is, in the direction of the center of the circle. So we have found both the magnitude and the direction of the acceleration for circular motion.

Notice how *unsophisticated* all of the steps have been! That is the great virtue of the method.

It is clear that Newton recognized from the symmetry of the problem that the *average* magnitude of the acceleration (as we just calculated it) would be the same as the *instantaneous* value of the same quantity. There is some virtue, however, in *directly calculating* the instantaneous value—in fact, some instructors may find this approach pedagogically preferable. With our same trigonometric substitution, the ''single bounce'' change in velocity yields $\Delta v = 2v \cos \theta \approx v\varphi$ for the case of many bounces (i.e., for the case that φ is small). If Δt is the time to go through (small) φ, then $\Delta t \approx \varphi r/v$ and the instantaneous acceleration

$$a = \frac{\Delta v}{\Delta t} = \frac{v\varphi}{\left(\dfrac{\varphi r}{v}\right)} = \frac{v^2}{r}, \tag{9}$$

which is equal to our previous value.

Huygens' method. Huygens considered a particle that is thrown sideways, which he knew would fall in a parabolic curve. He constructed the largest circle that passes through the original position of the particle, and yet does not cut the parabola. The method is *not* elegant. Newton, in his second approach to the same problem, like Huygens considered a comparison with motion under gravitation.

This alternate approach is presented elegantly and concisely by Tipler,[7] Tipler's exposition requiring no vector concepts. However, these proofs by Huygens, by Newton, and by Tipler, do mix the *general* concept of circular motion with the *particular* question of motion under gravity, and they also do require the student to know that for constant acceleration a, the distance traveled in time t, is $at^2/2$. In contrast, Newton's first method (which I have presented above) has *neither* of these complications, and is therefore strongly to be preferred for the student's first introduction to the acceleration that is associated with circular motion.

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[a]Deceased.

[b]Henry A. Rowland Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218; electronic mail: henry@jhu.edu

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[7]P. A. Tipler, *Physics for Scientists and Engineers* (Worth, New York, 1991), 3rd ed., p. 66.

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CLEANING THE BLACKBOARD

To further enhance our readiness, Fermi used his clout as a member of the Accademia d'Italia to promote a small international nuclear physics conference, which was held in Rome in October 1931 and attended by about thirty well-chosen physicists. At the conference I had the privilege of cleaning the blackboard for Marie Curie. Regrettably, I did not do it to her satisfaction, and she told me so in no uncertain terms.

Emilio Segrè, *A Mind Always in Motion—The Autobiography of Emilio Segrè* (University of California Press, Berkeley, 1993), p. 88.